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On an inverse thermo-elasticity problem for an infinite medium containing a cavity of unknown shape

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Abstract The static inverse thermoelastic problem for an infinite elastic isotropic medium containing a cavity of unknown shape, under three-axes tension and given constant values of the pressure and the temperature on the cavity surface, is considered. The shape of a cavity is sought subject to the condition that certain stress components are uniform on the cavity surface. It is shown that ellipsoidal shapes furnish a solution of this inverse thermo-elasticity problem. Nonlinear equations for determining the geometric cavity parameters, which lead to an equal-stress state along the cavity surface, are obtained. Results of other authors for force loading only are obtained as special cases. Numerical investigations have been carried out and correlations between values of the force and temperature loadings and geometrical parameters for an equal-stress cavity surface are studied. The stress values on the equal-stress cavity surface under force and temperature loadings are investigated.

Keywords Equal-stress surface · Force and temperature loading · Inverse problem · Unknown cavity surface

1 Introduction

Stress concentrations near holes, cavities and inclusions considerably influence the strength of design elements. The problem of finding the cavity or inclusion shape to minimize stress concentration under force and temperature influences is a difficult and important problem in the optimization of construction elements. Like other problems of shape optimization, the geometry of a cavity or inclusion is the unknown element. However, the quantity to be minimized differs in principle from the functionals of the integral form that are usually encountered in classical variational problems. The solution of this problem allows estimating the limiting possibilities to decrease stress concentration under a given system of loadings, and also to investigate the properties of such limiting states.

We consider the three-dimensional problem of finding a cavity shape in an infinite elastic medium under force and temperature loading. We find the cavity shape as a function of the loading parameters and of the properties of the elastic medium that cause the special stress distribution along the cavity surface. Similar two-dimensional inverse problems for finding the optimal hole shape in a plate under tension were considered in [1,2]. Some space inverse problems of finding the optimal cavity shape or rigid inclusion shape under three-axes tension were investigated in [3–8]. The solution of a more complex three-dimensional inverse problem for an infinite medium with an

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elastic isotropic inclusion, the elastic properties of which differ from the medium properties, under force loading, was obtained in [9]. Besides, it is assumed that between medium and inclusion ideal mechanical contact takes place. Inverse thermo-elasticity problems for an isotropic matrix with an elastic isotropic or transversally isotropic inclusion under three-axes tension and uniform heating of matrix and inclusion were studied in [10,11]. We note that some direct problems for ellipsoidal cavities and inclusions in anisotropic media, which could be used in some cases for solving inverse problems, were considered in [12–15], [16, Chapt. 4].

Note that a significant change of the temperature field can change the mechanical and thermal properties of a material. According to linear theory [17, Chapt. 13], we assume that a small change of the temperature field does not change the elastic and thermal properties of the material.

The problem of finding the cavity shape which realizes an equal-stress state along the cavity surface, under three-axes tension in the elastic isotropic medium and given values of pressure and temperature on the cavity surface, is considered in this paper. We note that the temperature field has a more complicated distribution in the medium in this case than in the above-mentioned papers, which were devoted to solving inverse problems. As in [3–10], the unknown cavity shape is found from the condition of equal-stress state along the cavity surface. If the temperature field in the elastic medium is absent, we have to deal with the inverse elastic problem only. This was studied in [8]. Its solution follows as a special case from the results obtained here.

2 The inverse thermo-elasticity problem formulation

Suppose that an elastic isotropic medium contains a cavity of unknown shape. Let $\sigma_{11}^{\infty}, \sigma_{22}^{\infty}, \sigma_{33}^{\infty}$ denote a given constant tensile stress in the medium, P_0 denote pressure on the cavity surface and T_0 denote the temperature on a cavity surface of unknown shape.

It is necessary to define a cavity shape which leads to the equal-stress state along its entire surface. For this, we consider the next thermo-elasticity problem:

$$\frac{1}{1-2\nu}\,\text{grad div}\vec{u} + \nabla^2\vec{u} = \frac{2(1+\nu)\alpha}{1-2\nu}\,\text{grad}\,T, \quad (\vec{x} \in R^3), \tag{1}$$

$$\sigma_{ij}\, n_j\big|_S = -P_0 n_i, \tag{2}$$

$$\sigma_{ij} = \sigma_{ij}^{\infty} + o(|\vec{x}|^{-1}) \quad \text{as } |\vec{x}| \to \infty, (i, j = 1, 2, 3), \tag{3}$$

$$\nabla^2 T = 0;\ T|_S = T_0, \quad T = O(|\vec{x}|^{-1}) \quad \text{as } |\vec{x}| \to \infty. \tag{4}$$

Note, that (1) follows easily from [17]. The cavity surface S is an unknown surface which we define from the condition that a special stress structure exists on its surface. For given values σ_{ij}^{∞}, P_0 and T_0, elastic matrix properties μ, ν and linear temperature extension coefficient α, we find such a cavity surface S for which the stress tensor has a special distribution along the cavity surface [8]:

$$\sigma_{ij}\big|_S = \sigma_S(\delta_{ij} - n_i n_j) - P_0 n_i n_j, \quad (i, j = 1, 2, 3), \tag{5}$$

where σ_S is an unknown value, which we have to find as part of the problem solution; δ_{ij} is the Kronecker symbol and n_i are the components of the unit outward normal to the cavity surface. We can verify that an equal-stress state is realized on the cavity surface for this kind of stress distribution. In curvilinear coordinates connected to the surface S, the stress state on this surface has the following form [8,9]:

$$\sigma_{nn}|_S = -P_0, \quad \sigma_{\tau\tau}|_S = \sigma_{tt}|_S = \sigma_S, \quad \sigma_{n\tau}|_S = \sigma_{nt}|_S = \sigma_{t\tau}|_S = 0.$$

Note that these conditions are equivalent to conditions (5), (this can be proved directly), and they can be considered as the equivalent criterion for finding an equal-stressed surface S.

3 Solution of the inverse thermo-elasticity problem

Using the knowledge of the solutions of inverse problems [2–11], for which the required geometrical form of a single cavity or inclusion has been found in a class of ellipsoidal shapes, we shall search for the unknown geometrical form of a cavity in the same class of geometrical shapes with unknown values of ellipsoidal semi-axes.

According to [18, Chapt. 5] we have the collection of harmonic functions

$$\omega_n(x, y, z, \rho) = \int_\rho^\infty \left(\frac{x_1^2}{a_1^2 + s} + \frac{x_2^2}{a_2^2 + s} + \frac{x_3^2}{a_3^2 + s} - 1 \right)^n \frac{\mathrm{d}s}{\left[(a_1^2 + s)(a_2^2 + s)(a_3^2 + s)\right]^{1/2}}, \quad (n = 0, 1, 2, 3, \ldots).$$

For $n = 0$ we obtain the harmonic function $\omega_0(x, y, z, \rho)$ which has a constant value on the ellipsoidal surface. The function $\omega_1(x, y, z, \rho)$ which we have for $n = 1$, corresponds to the Newtonian potential of a homogeneous ellipsoid of unit density [16, Chapt. 2], [19, Chapt. 5].

The temperature field in the elastic medium with cavity is represented by means of the harmonic function $\omega_0(x, y, z, \rho)$

$$T(x_1, x_2, x_3) = C \int_\rho^\infty \frac{\mathrm{d}s}{Q(s)}, \quad Q(s) = [(a_1^2 + s)(a_2^2 + s)(a_3^2 + s)]^{1/2}, \tag{6}$$

where the constant C is given by $C = T_0 / \int_0^\infty \mathrm{d}s/Q(s)$ and $\rho = \rho(x_1, x_2, x_3)$ is an ellipsoidal coordinate, which is connected with Cartesian coordinates by the relation

$$\frac{x_1^2}{a_1^2 + \rho} + \frac{x_2^2}{a_2^2 + \rho} + \frac{x_3^2}{a_3^2 + \rho} - 1 = 0, \tag{7}$$

Here the parameters a_1, a_2, a_3 are ellipsoidal semi-axes, which we need to determine during the solution process; on the cavity surface we have $\rho = 0$.

According to [12] we now write the particular solution of the thermo-elasticity equations in the following form

$$\vec{u}^* = D \operatorname{grad} \chi, \tag{8}$$

where

$$\chi = -\int_\rho^\infty \frac{(a_1^2 + s)}{Q(s)} \left[1 - \frac{x_1^2}{a_1^2 + s} - \frac{x_2^2}{a_2^2 + s} - \frac{x_3^2}{a_3^2 + s} \right] \mathrm{d}s. \tag{9}$$

Displacement components, which correspond to this particular solution of the thermo-elasicity equations, can be represented as (see [12]):

$$\begin{aligned} u_x^* &= 2Dx_1 \int_\rho^\infty \frac{\mathrm{d}s}{Q(s)}; \quad u_y^* = 2Dx_2 \int_\rho^\infty \frac{\mathrm{d}s}{Q(s)} + (a_1^2 - a_2^2) \int_\rho^\infty \frac{\mathrm{d}s}{(a_2^2 + s)Q(s)}; \\ u_z^* &= 2Dx_3 \int_\rho^\infty \frac{\mathrm{d}s}{Q(s)} + (a_2^2 - a_3^2) \int_\rho^\infty \frac{\mathrm{d}s}{(a_3^2 + s)Q(s)}, \end{aligned} \tag{10}$$

where the constant D is defined as follows:

$$D = \frac{1}{4} \frac{(1 + \nu)}{(1 - \nu)} \frac{\alpha T_0}{\int_0^\infty \frac{\mathrm{d}s}{Q(s)}}. \tag{11}$$

Using (10), we find expressions for the stress components which correspond to the particular equations of the thermo-elasticity equations (8)

$$\sigma_{11}^{*}=4D\left((\lambda+\mu)\int_{\rho}^{\infty}\frac{\mathrm{d}s}{Q(s)}-2\mu\frac{(a_1^2+\rho)}{Q(\rho)}\left(\frac{x_1}{(a_1^2+\rho)h}\right)^2\right),$$
$$\sigma_{22}^{*}=4D\left((\lambda+\mu)\int_{\rho}^{\infty}\frac{\mathrm{d}s}{Q(s)}+\mu(a_1^2-a_2^2)\int_{\rho}^{\infty}\frac{\mathrm{d}s}{(a_2^2+s)Q(s)}-2\mu\frac{(a_1^2+\rho)}{Q(\rho)}\left(\frac{x_2}{(a_2^2+\rho)h}\right)^2\right),$$
$$\sigma_{33}^{*}=4D\left((\lambda+\mu)\int_{\rho}^{\infty}\frac{\mathrm{d}s}{Q(s)}+\mu(a_1^2-a_3^2)\int_{\rho}^{\infty}\frac{\mathrm{d}s}{(a_3^2+s)Q(s)}-2\mu\frac{(a_1^2+\rho)}{Q(\rho)}\left(\frac{x_3}{(a_3^2+\rho)h}\right)^2\right), \tag{12}$$
$$\sigma_{12}^{*}=4D\left(-2\mu\frac{(a_1^2+\rho)}{Q(\rho)}\frac{x_1}{(a_1^2+\rho)h}\frac{x_2}{(a_2^2+\rho)h}\right),\quad \sigma_{13}^{*}=4D\left(-2\mu\frac{(a_1^2+\rho)}{Q(\rho)}\frac{x_1}{(a_1^2+\rho)h}\frac{x_3}{(a_3^2+\rho)h}\right),$$
$$\sigma_{23}^{*}=4D\left(-2\mu\frac{(a_1^2+\rho)}{Q(\rho)}\frac{x_2}{(a_2^2+\rho)h}\frac{x_3}{(a_3^2+\rho)h}\right),\quad h=\frac{x_1^2}{(a_1^2+\rho)^2}+\frac{x_2^2}{(a_2^2+\rho)^2}+\frac{x_3^2}{(a_3^2+\rho)^2}.$$

By means of (12), if $\rho=0$, we obtain the structure of the stress field on the cavity surface, which results from this particular solution of the thermo-elasticity equation. As a result we obtain

$$\sigma_{11}^{*}\big|_S=4D\left((\lambda+\mu)\int_{0}^{\infty}\frac{\mathrm{d}\lambda}{Q(\lambda)}-2\mu\frac{a_1^2}{a_1a_2a_3}n_1^2\right),$$
$$\sigma_{22}^{*}\big|_S=4D\left((\lambda+\mu)\int_{0}^{\infty}\frac{\mathrm{d}s}{Q(s)}+\mu(a_1^2-a_2^2)\int_{0}^{\infty}\frac{\mathrm{d}s}{(a_2^2+s)Q(s)}-2\mu\frac{a_1^2}{a_1a_2a_3}n_2^2\right),$$
$$\sigma_{33}^{*}\big|_S=4D\left((\lambda+\mu)\int_{0}^{\infty}\frac{\mathrm{d}s}{Q(s)}+\mu(a_1^2-a_3^2)\int_{0}^{\infty}\frac{\mathrm{d}s}{(a_3^2+s)Q(s)}-2\mu\frac{a_1^2}{a_1a_2a_3}n_3^2\right), \tag{13}$$
$$\sigma_{12}^{*}\big|_S=4D\left(-2\mu\frac{a_1^2}{a_1a_2a_3}n_1n_2\right),\quad \sigma_{13}^{*}\big|_S=4D\left(-2\mu\frac{a_1^2}{a_1a_2a_3}n_1n_3\right),\quad \sigma_{23}^{*}\big|_S=4D\left(-2\mu\frac{a_1^2}{a_1a_2a_3}n_2n_3\right).$$

Comparing the obtained expressions (13) with (5), we may conclude that the multiplier at n_in_j is identical in all expressions $\sigma_{ij}^{*}|_S (i\neq j)$ in (13).

We now represent the stress state in the medium with cavity as a superposition of states:

$$\sigma_{ij}=\sigma_{ij}^{0}+\sigma_{ij}^{*}+\sigma_{ij}^{\infty}-2\mu\alpha T\frac{(1+\nu)}{(1-2\nu)}\delta_{ij}. \tag{14}$$

In addition, the solution of the uniform equilibrium equations σ_{ij}^{0} is represented according to [8] as

$$\sigma_{ij}^{0}=\frac{A}{4\pi}\frac{\partial^2\Omega}{\partial x_i\partial x_j},\quad \Omega=\pi a_1a_2a_3\int_{\rho}^{\infty}\left(\frac{x_1^2}{a_1^2+s}+\frac{x_2^2}{a_2^2+s}+\frac{x_3^2}{a_3^2+s}-1\right)\frac{\mathrm{d}s}{Q(s)}. \tag{15}$$

Next compare coefficients for similar terms. After some transformations which are similar to those of [8], we obtain that the conditions (5) on the special structure of the stress tensor on the cavity surface are realized if a system of the four following equations with respect to the unknown $A, \sigma_s, a_2/a_1, a_3/a_1$ is satisfied. These equations are linear relative to the first two unknown values and nonlinear with respect to the last two ones.

$$\sigma_S+P_0=A+\left(\frac{1+\nu}{1-\nu}\right)\alpha\mu T_0\frac{a_1^2}{J_0(0)},$$

$$A\left(J_1(0)-J_2(0)\right)+\left(\sigma_{11}^{\infty}-\sigma_{22}^{\infty}\right)+\left(\frac{1+\nu}{1-\nu}\right)\alpha\mu T_0\left(\frac{\left(a_2^2-a_1^2\right)J_2(0)}{J_0(0)}\right)=0,$$

$$A\left(J_1(0)-J_3(0)\right)+\left(\sigma_{11}^{\infty}-\sigma_{33}^{\infty}\right)+\left(\frac{1+\nu}{1-\nu}\right)\alpha\mu T_0\left(\frac{\left(a_3^2-a_1^2\right)J_2(0)}{J_0(0)}\right)=0, \tag{16}$$

$$3\sigma_S=A+\left(\sigma_{11}^{\infty}+\sigma_{22}^{\infty}+\sigma_{33}^{\infty}\right)-3\alpha\mu T_0\left(\frac{1+\nu}{1-\nu}\right)+\alpha\mu T_0\left(\frac{1+\nu}{1-\nu}\right)\left(\frac{a_1^2}{J_0(0)}-1\right),$$

where

$$J_0(\rho)=\frac{1}{2}a_1a_2a_3\int_{\rho}^{\infty}\frac{\mathrm{d}s}{Q(s)},\quad J_i(\rho)=\frac{1}{2}a_1a_2a_3\int_{\rho}^{\infty}\frac{\mathrm{d}s}{\left(a_i^2+s\right)Q(s)},\quad (i=1,2,3). \tag{17}$$

As follows from formulas (17), the values $J_0(0)$, $J_1(0)$, $J_2(0)$, $J_3(0)$ are functions of ellipsoid semi-axes. For the further calculations, it is convenient to use their interconnection in accordance with [16, Chapt. 2].

$$J_0(0)=a_1^2J_1(0)+a_2^2J_2(0)+a_3^2J_3(0).$$

Further, from system of equations (16), we find

$$\begin{aligned}\sigma_S&=\frac{\sigma_{11}^{\infty}+\sigma_{22}^{\infty}+\sigma_{33}^{\infty}}{2}+\frac{P_0}{2}-2\mu\alpha T_0\left(\frac{1+\nu}{1-\nu}\right);\\ A&=\frac{\sigma_{11}^{\infty}+\sigma_{22}^{\infty}+\sigma_{33}^{\infty}}{2}+\frac{3}{2}P_0-\mu\alpha T_0\left(\frac{1+\nu}{1-\nu}\right)\left(2+\frac{a_1^2}{J_0(0)}\right).\end{aligned} \tag{18}$$

The final two equations can be rewritten as

$$\begin{aligned}&\left[\frac{1}{2}\left(1+\frac{\sigma_{22}^{\infty}}{\sigma_{11}^{\infty}}+\frac{\sigma_{33}^{\infty}}{\sigma_{11}^{\infty}}\right)+\frac{3}{2}\frac{P_0}{\sigma_{11}^{\infty}}-\frac{\mu\alpha T_0}{\sigma_{11}^{\infty}}\left(\frac{1+\nu}{1-\nu}\right)\left(2+\frac{a_1^2}{J_0(0)}\right)\right](J_1(0)-J_2(0))+\left(1-\frac{\sigma_{22}^{\infty}}{\sigma_{11}^{\infty}}\right)\\ &\quad+\frac{\mu\alpha T_0}{\sigma_{11}^{\infty}}\left(\frac{1+\nu}{1-\nu}\right)\left(\frac{(a_2^2-a_1^2)J_2(0)}{J_0(0)}\right)=0,\\ &\left[\frac{1}{2}\left(1+\frac{\sigma_{22}^{\infty}}{\sigma_{11}^{\infty}}+\frac{\sigma_{33}^{\infty}}{\sigma_{11}^{\infty}}\right)+\frac{3}{2}\frac{P_0}{\sigma_{11}^{\infty}}-\frac{\mu\alpha T_0}{\sigma_{11}^{\infty}}\left(\frac{1+\nu}{1-\nu}\right)\left(2+\frac{a_1^2}{J_0(0)}\right)\right](J_1(0)-J_3(0))+\left(1-\frac{\sigma_{33}^{\infty}}{\sigma_{11}^{\infty}}\right)\\ &\quad+\frac{\mu\alpha T_0}{\sigma_{11}^{\infty}}\left(\frac{1+\nu}{1-\nu}\right)\left(\frac{(a_3^2-a_1^2)J_3(0)}{J_0(0)}\right)=0.\end{aligned} \tag{19}$$

Equations (19), which connect a given force and temperature loading and medium properties with geometrical cavity parameters, are parametrically nonlinear equations with respect to the parameters a_2/a_1 and a_3/a_1.

4 Numerical results and their analysis

We now analyze Eqs. (18), (19). Theoretically, when a temperature influence is absent, the corresponding relations from [8] should be obtained from these formulas as special cases. Indeed, if $T_0=0$, we obtain

$$\begin{aligned}&\sigma_S=\frac{\sigma_{11}^{\infty}+\sigma_{22}^{\infty}+\sigma_{33}^{\infty}}{2}+\frac{P_0}{2},\quad A=\frac{\sigma_{11}^{\infty}+\sigma_{22}^{\infty}+\sigma_{33}^{\infty}}{2}+\frac{3}{2}P_0,\\ &J_2(0)-J_1(0)=\frac{\sigma_{11}^{\infty}-\sigma_{22}^{\infty}}{A},\quad J_3(0)-J_1(0)=\frac{\sigma_{11}^{\infty}-\sigma_{33}^{\infty}}{A}.\end{aligned} \tag{20}$$

These expressions coincide entirely with the results of [8], where a similar problem for the action of force loading is considered.

Now we consider the case of an axially symmetric tension $\sigma_{22}^{\infty}=\sigma_{11}^{\infty}$ for given values of the pressure P_0 and the temperature T_0. It is obvious that for an elastic isotropic medium a search of the necessary geometrical form of a cavity must be carried out within a class of rotational forms for reasons of symmetry. Indeed, it is easy to verify that the first equation of system (19) is satisfied if $a_1 = a_2$. In this case, from system of nonlinear equations (19), we obtain a single nonlinear equation with regard to a_3/a_1. This means that the unknown cavity shape is searched already among rotationally symmetric ellipsoids (spheroids).

Let us note an important property of system (16) which essentially simplifies our search. For a given ratio of the ellipsoid semi-axes and values $\mu\alpha T_0/\sigma_{11}^{\infty}$ and P_0/σ_{11}^{∞}, this system is linear with respect to the unknown ratios of loadings $\sigma_{22}^{\infty}/\sigma_{11}^{\infty}$ and $\sigma_{33}^{\infty}/\sigma_{11}^{\infty}$, which lead to the realization of an equal-stress state for this ellipsoidal cavity shape.

This property allows estimating the value of a payoff by reduction of the stress concentration in comparison, for example, with the most symmetric, namely the spherical form of the cavity. So, if $T_0 = 0$, the values of the two ellipsoidal semi-axes are fixed and the third semi-axis is changed, making it is possible to calculate the loadings realizing an equal-stress state for obtaining the geometry of the cavity. Then, it is possible to calculate the greatest stress value on the spherical cavity (σ_{sphere}) under the same loadings and to estimate the value of a payoff by a reduction of the stress concentration for the equal-stress shape in comparison with the spherical cavity shape. As a result of our numerical calculations, we obtained that for $a_3/a_1 = 0.1$ and $a_2/a_1 = 0.2, 0.4, 0.6, 0.8$, the magnitudes of the payoffs are equal $\sigma_{\text{sphere}}/\sigma_S = 1.8314, 1.9001, 1.9218, 1.9305$, respectively.

In some cases (under specially chosen loadings), the stress concentration on the equal-stress cavity surface is significantly less (almost twice) than for a spherical cavity.

The relations between the stress and the semi-axes parameters of the equal-stress shape under three-axes tension ($T_0 = P_0 = 0$) are shown in Figs. 1–3.

In these figures the curves 1–5 correspond to semi-axes ratios of ellipsoids $a_2/a_1 = 1, 0.8, 0.6, 0.4, 0.2$, respectively. The line 1 relates to the case of axially symmetric loading. In Figs. 1–2 the correlation between loading ratios that lead to the equal-stress state, and the semi-axes of equal-stress cavity are shown. Figure 3 demonstrates the dependence of the stress values on the equal-stress shape of the cavity semi-axes. The influence of the temperature action on the parameters of the equal stress-state is shown in Figs. 4–6. In addition, it was assumed that $a_2/a_1 = 0.6$ and the absence of pressure on the surfaces of a cavity ($P_0 = 0$). Curve 1 in these figures corresponds to the case $\beta = \alpha\mu T_0/\sigma_{11}^{\infty} = 0$ (force loading only), and lines 2, 3, 4 relate to the cases $\beta = 0.15, 0.25, 0.35$, respectively.

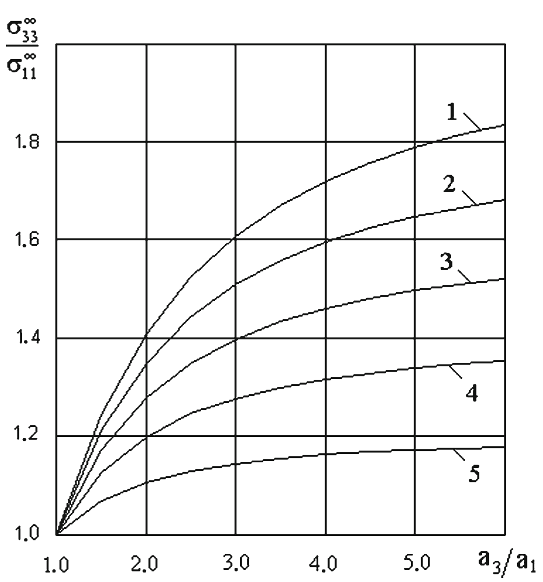


Fig. 1 The dependence of $\sigma_{33}^{\infty}/\sigma_{11}^{\infty}$ on the semi-axis ratio of an equal-stress ellipsoidal cavity under three-axes tension

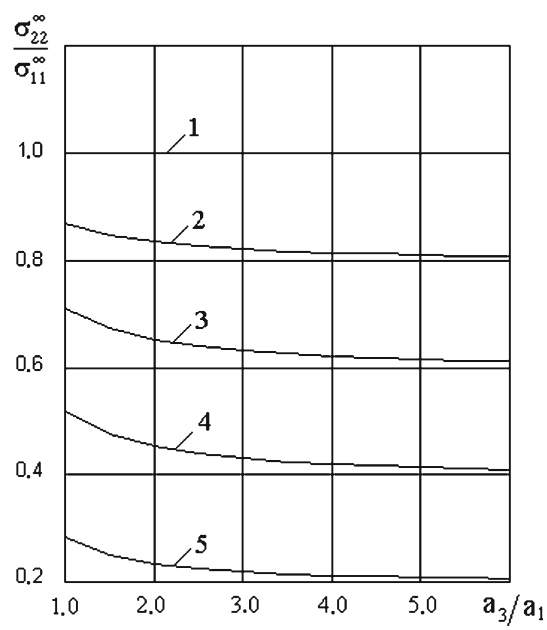


Fig. 2 The dependence of $\sigma_{22}^{\infty}/\sigma_{11}^{\infty}$ on the semi-axis ratio of an equal-stress ellipsoidal cavity under three-axes tension

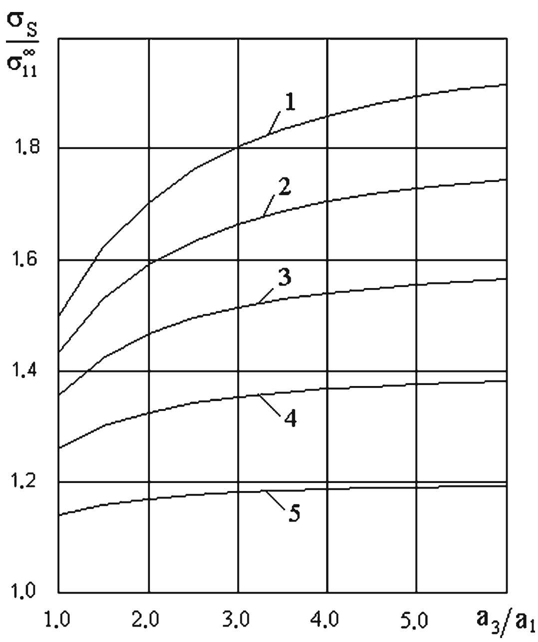


Fig. 3 The dependence of the maximal stress on the semi-axis ratio of an equal-stress ellipsoidal cavity under three-axes tension

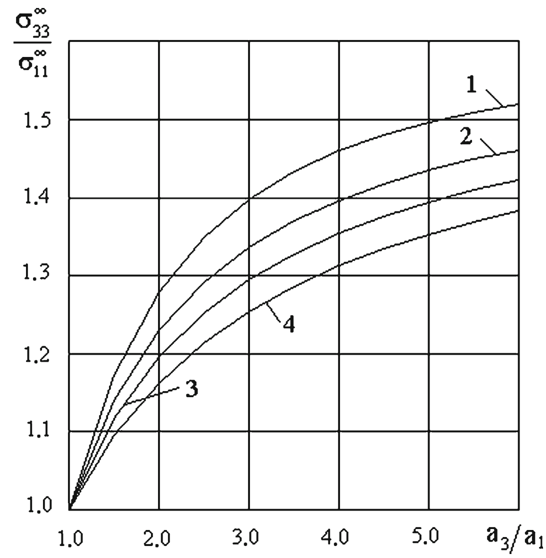


Fig. 4 The dependence of $\sigma_{22}^{\infty}/\sigma_{11}^{\infty}$ on the semi-axis ratio of an equal-stress ellipsoidal cavity surface under three-axes tension and temperature influence

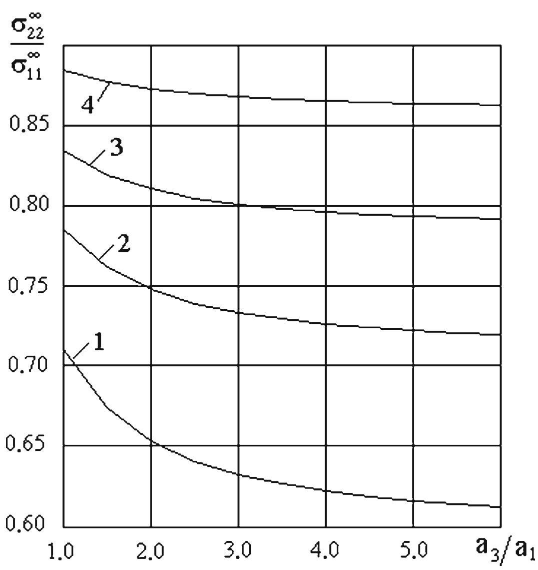


Fig. 5 The dependence of $\sigma_{22}^{\infty}/\sigma_{11}^{\infty}$ on the semi-axis ratio of an equal-stress ellipsoidal cavity surface under three-axes tension and temperature influence

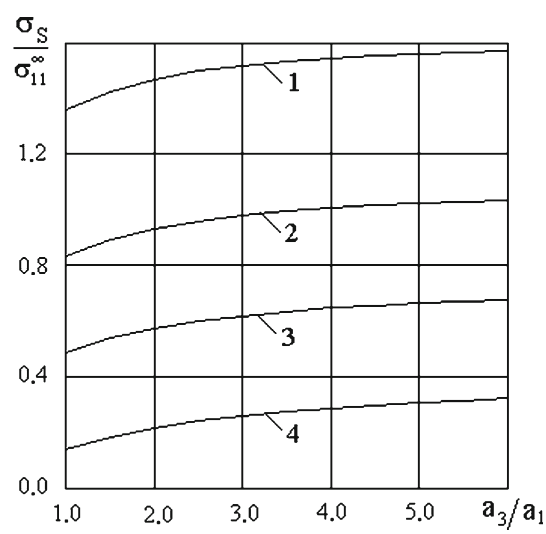


Fig. 6 The dependence of the maximal stress values on an equal-stress ellipsoidal cavity surface on the semi-axis ratio under three-axes tension and temperature influence

In these figures, we can estimate the influence of a temperature field on the interrelation between loadings and geometrical parameters of equal-stress cavities, and also on the value of the stress on the surface of equal-stress cavities. Results of investigations of the pressure influence on the parameters of the equal-stress state are shown in Figs. 7–9. The ratio of the semi-axes was $a_2/a_1 = 0.6$ (as well as in the above mentioned case) and $T_0 = 0$.

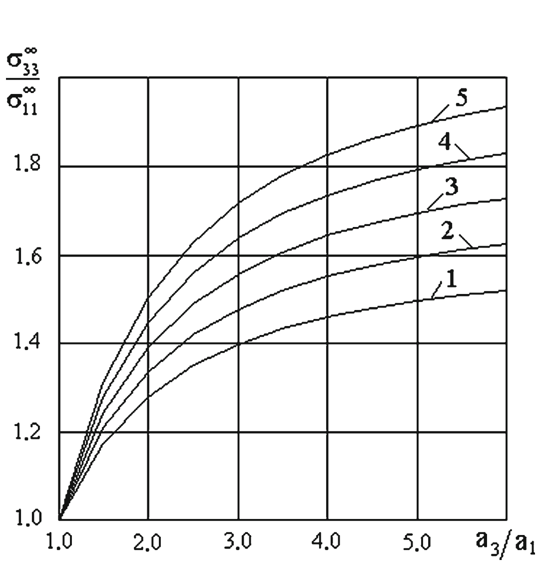


Fig. 7 The dependence of $\sigma_{33}^{\infty}/\sigma_{11}^{\infty}$ on the semi-axis ratio of an equal-stress ellipsoidal cavity surface under three-axes tension and inner pressure on the cavity surface

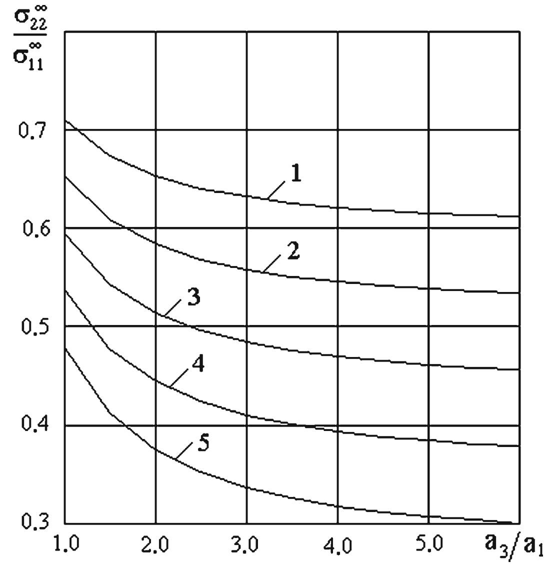
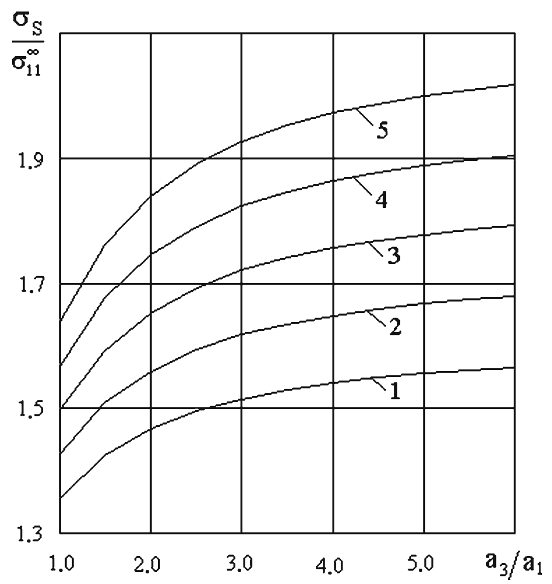


Fig. 8 The dependence of $\sigma_{22}^{\infty}/\sigma_{11}^{\infty}$ on the semi-axis ratios of an equal-stress ellipsoidal cavity under three-axes tension and inner pressure on the cavity surface

Fig. 9 The dependence of the maximal stress values on the semi-axis ratio of an equal-stress ellipsoidal cavity under three-axes tension and inner pressure on the cavity surface



Curves 1–5 correspond to the cases $P_0/\sigma_{11}^{\infty} = 0$, 0.2, 0.4, 0.6, 0.8, respectively. For all numerical calculations, the Poisson ratio of the elastic medium was $\nu = 0.3$. We observe the correlation between the ratios of loadings and semi-axes of ellipsoidal equal-stress cavities, and can estimate the stress values on the equal-stress cavity surface.

5 Conclusions

The inverse problem of thermo-elasticity for an isotropic medium of unknown form of a cavity under force, temperature loading and inner pressure has been considered. The nonlinear equations for searching the geometrical parameters of an equal-stress cavity surface have been obtained. The relations for the force loadings which were found by other authors follow from the obtained equations as special cases. Numerical calculations that establish the correspondence between loads and geometrical parameters of the required cavity shape were carried out and stress values at the surface of an equal-stress cavity were obtained. The influence of the temperature and the inner pressure on the relation between the loading parameters and the geometrical characteristics of an equal-stress cavity was evaluated.

It was shown that in some cases the obtained equal-stress cavity shapes allow to decrease the stress concentration notably. A temperature field and a pressure upon the cavity surface, as well as the force loading values, strongly influence the geometrical parameters of an equal-stress cavity and the stress values on its surface.

Future researches may be devoted to studying similar inverse problems for anisotropic two-dimensional and three-dimensional elastic bodies with cavities or inclusions of unknown form under given force loading and temperature influences. This will allow evaluating opportunities on how to decrease stress concentration in anisotropic bodies.

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Appendix A

According to [16, Chapt. 2]; [20, Chapt. 17] the integrals $J_0(0), J_1(0), J_2(0), J_3(0)$ can be expressed by standard elliptical integrals. Assuming $a_1 > a_2 > a_3$, we have:

$$J_0(0) = \frac{a_1a_2a_3}{\left(a_1^2-a_3^2\right)^{1/2}}F(\theta,k), \quad J_1(0) = \frac{a_1a_2a_3}{\left(a_1^2-a_2^2\right)\left(a_1^2-a_3^2\right)^{1/2}}\{F(\theta,k)-E(\theta,k)\}, \tag{A.1}$$

$$J_2(0) = a_1a_2a_3\left\{\frac{\left(a_1^2-a_3^2\right)^{1/2}}{\left(a_1^2-a_2^2\right)\left(a_2^2-a_3^2\right)}E(\theta,k) - \frac{F(\theta,k)}{\left(a_1^2-a_2^2\right)\left(a_1^2-a_3^2\right)^{1/2}} - \frac{a_3}{a_1a_2\left(a_2^2-a_3^2\right)}\right\}, \tag{A.2}$$

$$J_3(0) = \frac{a_1a_2a_3}{\left(a_2^2-a_2^2\right)\left(a_1^2-a_3^2\right)^{1/2}}\left\{\frac{a_2\left(a_1^2-a_3^2\right)^{1/2}}{a_1a_3} - E(\theta,k)\right\}, \tag{A.3}$$

$$a_1^2J_1(0) + a_1^2J_2(0) + a_3^2J_3(0) = \frac{a_1a_2a_3}{\left(a_1^2-a_3^2\right)^{1/2}}F(\theta,k) = J_0(0), \quad J_1(0)+J_2(0)+J_3(0) = 1, \tag{A.4}$$

where

$$F(\theta,k) = \int_0^\theta \frac{\mathrm{d}w}{\left(1-k^2\sin^2 w\right)^{1/2}}, \quad E(\theta,k) = \int_0^\theta \left(1-k^2\sin^2 w\right)^{1/2}\mathrm{d}w,$$

$$\theta = \sin^{-1}\left(1-a_3^2/a_1^2\right)^{1/2}, \quad k = \left\{\left(a_1^2-a_2^2\right)/\left(a_1^2-a_3^2\right)\right\}^{1/2}.$$

Appendix B

Consider the superposition of stress states according to (14). We shall equate each of the stress-state components of this superposition to the required value according to (5). We have

$$\sigma_{11}|_S = A(J_1(0)-n_1^2) + \sigma_{11}^\infty + 4D\left((\lambda+\mu)J_0(0) - 2\mu\frac{a_1^2}{a_1a_2a_3}n_1^2\right) - 2\mu\alpha T_0\left(\frac{1+\nu}{1-2\nu}\right) = \sigma_S - (\sigma_S+P_0)n_1^2,$$

$$\sigma_{22}|_S = A(J_2(0) - n_2^2) + \sigma_{22}^{\infty} + 4D\left((\lambda+\mu)J_0(0) + \mu(a_1^2 - a_2^2)J_2(0) - 2\mu\frac{a_1^2}{a_1a_2a_3}n_2^2\right)$$
$$-2\mu\alpha T_0\left(\frac{1+\nu}{1-2\nu}\right) = \sigma_S - (\sigma_S + P_0)n_2^2,$$
$$\sigma_{33}|_S = A(J_3(0) - n_3^2) + \sigma_{33}^{\infty} + 4D\left((\lambda+\mu)J_0(0) + \mu(a_1^2 - a_3^2)J_3(0) - 2\mu\frac{a_1^2}{a_1a_2a_3}n_3^2\right)$$
$$-2\mu\alpha T_0\left(\frac{1+\nu}{1-2\nu}\right) = \sigma_S - (\sigma_S + P_0)n_3^2,$$
$$\sigma_{12}|_S = -\left(A + 8D\mu\frac{a_1^2}{a_1a_2a_3}\right)n_1n_2 = -(\sigma_S + P_0)n_1n_2,$$
$$\sigma_{13}|_S = -\left(A + 8D\mu\frac{a_1^2}{a_1a_2a_3}\right)n_1n_3 = -(\sigma_S + P_0)n_1n_3,$$
$$\sigma_{23}|_S = -\left(A + 8D\mu\frac{a_1^2}{a_1a_2a_3}\right)n_2n_3 = -(\sigma_S + P_0)n_2n_3,$$

where D is defined according to (11). We have from last three equations

$$A + 8D\mu\frac{a_1^2}{a_1a_2a_3} = \sigma_S + P_0. \tag{B.1}$$

By means of (B.1) we obtain from first three equations

$$AJ_1(0) + \sigma_{11}^{\infty} + 4D(\lambda+\mu)J_0(0) - 2\mu\alpha T_0\left(\frac{1+\nu}{1-2\nu}\right) = \sigma_S, \tag{B.2}$$
$$AJ_2(0) + \sigma_{22}^{\infty} + 4D(\lambda+\mu)J_0(0) + 4D\mu\left(a_1^2 - a_2^2\right)J_2(0) - 2\mu\alpha T_0\left(\frac{1+\nu}{1-2\nu}\right) = \sigma_S, \tag{B.3}$$
$$AJ_3(0) + \sigma_{33}^{\infty} + 4D(\lambda+\mu)J_0(0) + 4D\mu\left(a_1^2 - a_3^2\right)J_3(0) - 2\mu\alpha T_0\left(\frac{1+\nu}{1-2\nu}\right) = \sigma_S. \tag{B.4}$$

Let us transform (B.2)–(B.4). Instead of (B.2) we shall take the difference of (B.2) and (B.3), instead of (B.3) we shall take the difference of (B.2) and (B.4), instead of (B.4) we shall take the sum of (B.2), (B.3), (B.4). Substituting D according to (11), we obtain

$$A\left(J_1(0) - J_2(0)\right) + (\sigma_{11}^{\infty} - \sigma_{22}^{\infty}) + \left(\frac{1+\nu}{1-\nu}\right)\alpha\mu T_0\left(\frac{(a_2^2 - a_1^2)J_2(0)}{J_0(0)}\right) = 0, \tag{B.5}$$
$$A\left(J_1(0) - J_3(0)\right) + (\sigma_{11}^{\infty} - \sigma_{33}^{\infty}) + \left(\frac{1+\nu}{1-\nu}\right)\alpha\mu T_0\left(\frac{(a_3^2 - a_1^2)J_3(0)}{J_0(0)}\right) = 0, \tag{B.6}$$
$$3\sigma_S = A + (\sigma_{11}^{\infty} + \sigma_{22}^{\infty} + \sigma_{33}^{\infty}) + 3\mu\alpha T_0\left(\frac{1+\nu}{1-\nu}\right)\frac{1}{(1-2\nu)} \tag{B.7}$$
$$+\mu\left(\frac{1+\nu}{1-\nu}\right)\alpha T_0\left(\frac{a_1^2}{J_0(0)} - 1\right) - 6\mu\alpha T_0\left(\frac{1+\nu}{1-2\nu}\right) = 0, \tag{B.8}$$

Substituting D in (B.1), we have

$$\sigma_S + P_0 = A + \mu\alpha T_0\left(\frac{1+\nu}{1-\nu}\right)\frac{a_1^2}{J_0(0)}. \tag{B.9}$$

From (B.7), (B.8) we obtain

$$\sigma_S = \frac{\sigma_{11}^{\infty} + \sigma_{22}^{\infty} + \sigma_{33}^{\infty}}{2} + \frac{P_0}{2} - 2\mu\alpha T_0 \left(\frac{1+\nu}{1-\nu}\right), \tag{B.10}$$

$$A = \frac{\sigma_{11}^{\infty} + \sigma_{22}^{\infty} + \sigma_{33}^{\infty}}{2} + \frac{3P_0}{2} - \mu\alpha T_0 \left(\frac{1+\nu}{1-\nu}\right)\left(2 + \frac{a_1^2}{J_0(0)}\right). \tag{B.11}$$

Substituting the expressions σ_S, A in (B.5) and (B.6), we obtain the equations (19).

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